Inscribed and circumscribed quadrilaterals

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If you don’t know how to start with a geometric problem, try to find four points lying on a common circle. This advice is quite old, but still very useful in the IMO problems. It has been used since the very beginning of the history of the Mathematical Olympiads and is also effectively applied nowadays. An observation that the four points are concyclic is often based on the following simple theorems coming from the school geometry.

Theorem 1.
Let $ABCD$ be a convex quadrilateral. Then the points $A$, $B$, $C$, $D$ lie on a common circle if and only if $\angle ABC + \angle CDA = 180^\circ$ (fig. 1).

Theorem 2.
Let $ABCD$ be a convex quadrilateral. Then the points $A$, $B$, $C$, $D$ lie on a common circle if and only if $\angle ACB = \angle ADB$ (fig. 2).

The following example is related to Problem 5 of the 1-st IMO (1959).

Example
Point $E$ lies on the side $BC$ of a square $ABCD$. Let $BFGE$ be a square lying outside of the square $ABCD$ (fig. 3). Prove that the lines $AE$, $CF$ and $DG$ intersect in a common point.

Solution
Let $P$ be the intersection of the lines $AE$ and $CF$ (fig. 4). Our goal is to prove that the line $DG$ passes through $P$. This will be achieved by observing that the measures of the angles $DPA$, $APF$ and $FPG$ sum up to $180^\circ$.

Note that from the equalities $AB = BC$, $\angle ABE = 90^\circ = \angle CBF$ and $BE = BF$ it follows that triangles $ABE$ and $CBF$ are congruent. Therefore $\angle BAP = \angle BCP$, which by Theorem 2 implies that the points $A$, $B$, $C$ and $P$ are concyclic. On the other hand, since $ABCD$ is a square, the circumcircle of triangle $ABC$ passes through the point $D$. It means that the five points $A$, $B$, $C$, $D$ and $P$ lie on a common circle.

Therefore $\angle CPA = \angle CBA = 90^\circ$, which gives $\angle EPF = 90^\circ = \angle EGF$. Theorem 2 yields that the points $E$, $P$, $G$, $F$ are concyclic. Since $EBFG$ is a square, the circumcircle of triangle
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**Theorem 3.**

From a point \( P \) lying outside of a circle \( \omega \) two tangents \( PA \) and \( PB \) are drawn (fig. 5). Then \( PA = PB \).

![fig. 5](image)

**Proof**

Let \( O \) be the center of \( \omega \) (fig. 6). Then by the Pythagoras Theorem we get

\[
PA^2 = PO^2 - OA^2 = PO^2 - OB^2 = PB^2,
\]

implying \( PA = PB \), which finishes the proof.

**Theorem 4.**

Assume the incircle of triangle \( ABC \) is tangent to the sides \( BC, CA \) and \( AB \) at \( D, E \) and \( F \), respectively. Set \( a = BC, b = CA, c = AB \). Then we have

\[
AE = AF = s - a, \quad BF = BD = s - b, \quad CD = CE = s - c,
\]

where \( s \) denotes the semiperimeter of triangle \( ABC \).

![fig. 7](image)

**Solution**

Let \( x = AE = AF, y = BF = BD, z = CD = CE \). Then we obtain the following system of equations:

\[
\begin{align*}
x + y &= c \\
y + z &= a \\
z + x &= b
\end{align*}
\]

Solving this system we get \( x = (b + c - a)/2, y = (c + a - b)/2, z = (a + b - c)/2 \), which immediately gives \( x = s - a, y = s - b, z = s - c \).

**Theorem 5.**

Let \( ABCD \) be a convex (fig. 8) or a concave (fig. 9) quadrilateral. Then there exists a circle inscribed in the quadrilateral \( ABCD \) if and only if \( AB + CD = BC + DA \).
**Proof**

We present a proof for a concave quadrilateral $ABCD$. The proof for the convex case is very similar.

Assume first that there exists a circle inscribed in a concave quadrilateral $ABCD$ and let this circle be tangent to the lines $AB, BC, CD, DA$ at $K, L, M, N$, respectively (fig. 10). Then applying Theorem 3 several times we obtain $AK = AN, BK = BL, DM = DN$ and $CM = CL$. Therefore

$$AB + CD = AK + BK + DM - CM = AN + BL + DN - CL = AD + BC.$$ 

Conversly, assume that in a concave quadrilateral $ABCD$ it holds $AB + CD = BC + AD$. Without loss of generality assume that $AB > BC$. Then the last equality implies that $AD > CD$.

Let $P$ be the point lying on the side $AB$ such that $BP = BC$ (fig. 11). Similarly, let $Q$ be the point lying on the side $AD$ such that $QD = CD$. Then

$$AP = AB - BP = AB - BC = AD - CD = AD - DQ = AQ.$$ 

Therefore the angle bisectors of the angles $ABC, CDA$ and $DAB$ are the perpendicular bisectors of the sides of triangle $CPQ$ and hence they intersect in a common point $I$. Moreover, the distances from the point $I$ to all the lines $AB, BC, CD, DA$ are equal, say $r$. Then the circle $\omega$ with center $I$ and radius $r$ lies inside the quadrilateral $ABCD$ and is tangent to all the lines $AB, BC, CD, DA$. Hence the circle $\omega$ is inscribed in a quadrilateral $ABCD$.

**Example**

Point $P$ lies inside a triangle $ABC$. The lines $AP, BP, CP$ intersect the sides $BC, CA, AB$ at $D, E, F$, respectively. Prove that if it is possible to inscribe circles in quadrilaterals $AFPE$ and $BDPF$, then it is also possible to inscribe a circle in the quadrilateral $CDPE$.

**Solution**

The main idea is to note that the two given circles are inscribed in concave quadrilaterals $ABPC$ and $ABCP$, which by Theorem 5 implies that $AC + BP = AB + CP$ and $AB + CP = BC + AP$. Hence $AC + BP = BC + AP$, which using again Theorem 5 implies that it is possible to inscribe a circle in a concave quadrilateral $APBC$. Therefore there exists a circle inscribed in a (convex) quadrilateral $CDPE$. 

![fig. 10](image1.png)

![fig. 11](image2.png)

![fig. 12](image3.png)
Problems

1. Let $ABC$ be an acute-angled triangle with $\angle ACB = 60^\circ$ (fig. 1). Points $D$ and $E$ are the feet of the perpendiculars from $A$ and $B$ to the lines $BC$ and $AC$, respectively. Point $M$ is the midpoint of the side $AB$. Prove that the triangle $DEM$ is equilateral.

2. In an acute-angled triangle $ABC$ points $D$, $E$, $F$ are the feet of the altitudes taken from the vertices $A$, $B$, $C$, respectively (fig. 2).
   (a) Prove that $DA$, $EB$, $FC$ are the angle-bisectors of the angles of triangle $DEF$.
   (b) Knowing that the measures of the angles of triangle $ABC$ are equal $45^\circ$, $60^\circ$, $75^\circ$, determine measures of the angles of triangle $DEF$.

3. Let $ABCD$ be a square (fig. 3). Points $E$ and $F$ lie on the sides $AB$ and $AD$, respectively, such that $\angle ECF = 45^\circ$. The diagonal $BD$ meets the lines $CE$ and $CF$ at $P$ and $Q$, respectively. Prove that the points $A$, $E$, $F$, $P$ and $Q$ are concyclic.

4. Given is an acute-angled triangle $ABC$. On sides $BC$ and $AC$ squares $BCFE$ and $ACGH$ are outwardly constructed (fig. 4). Prove that the lines $AF$, $BG$ and $EH$ intersect in a common point.

5. On sides $BC$, $CA$ and $AB$ of triangle $ABC$ three equilateral triangles $BCD$, $CAE$ and $ABF$ are constructed outwardly. (fig. 5). Prove that:
   (a) $AD = BE = CF$.
   (b) The lines $AD$, $BE$ and $CF$ intersect in a common point.

6. Point $C$ lies on the line segment $AB$. Equilateral triangles $BCD$, $CAE$ and $ABF$ are constructed, as shown on figure 6. Prove that the lines $AD$, $BE$, $CF$ intersect in a common point.
7. Let $ABC$ be a triangle such that $AC = BC$ (fig. 7). Point $M$ is the midpoint of the side $AB$. Point $D$ lies on the line segment $CM$. Let $K$ and $L$ be the feet of the perpendiculars from $D$ and $C$ onto $BC$ and $AD$, respectively. Prove that the points $K$, $L$ and $M$ are collinear.

8. Let $ABC$ be a triangle with $AC = BC$ (fig. 8). Point $M$ is the midpoint of $AB$ and point $D$ is the midpoint of $CM$. Let $S$ be the foot of the perpendicular from $M$ to $AD$. Prove that $BS$ and $CS$ are perpendicular.

9. Let $ABC$ be a triangle with $AC = BC$ (fig. 9). Point $M$ is the midpoint of $AB$. Point $P$ lies inside triangle $ABC$ and satisfies $\angle PAB = \angle PBC$. Prove that $\angle APM + \angle BPC = 180^\circ$.

10. In a convex quadrilateral $ABCD$ the following equalities hold (fig. 10)

$$\angle ADB = 2\angle ACB \quad \text{and} \quad \angle BDC = 2\angle BAC.$$ 

Prove that $AD = CD$.

11. Point $P$ lies inside a parallelogram $ABCD$ and satisfies the equalities (fig. 11)

$$\angle DAP = 2\angle PCD \quad \text{and} \quad \angle CBP = 2\angle CDP.$$ 

Prove that $AP = BP = BC$.

12. Points $E$ and $F$ lie on the sides $AB$ and $BC$ of a square $ABCD$, respectively, such that $BE = BF$ (fig. 12). Point $S$ is the foot of the perpendicular from $B$ to $CE$. Prove that $\angle DSF = 90^\circ$. 
13. Point $E$ lies on the side $BC$ of a square $ABCD$ (fig. 13). Points $P$ and $Q$ are the feet of perpendiculars from $E$ and $B$ to $BD$ and $DE$, respectively. Prove that the points $A$, $P$, $Q$ are collinear.

14. Point $P$ lies inside a parallelogram $ABCD$, such that $\angle PAB = \angle PCB$ (fig. 14). Prove that $\angle PBA = \angle PDA$.

15. Point $P$ lies outside of a parallelogram $ABCD$, such that $\angle PAB = \angle PCB$ (fig. 15). Prove that $\angle APB = \angle CPD$.

16. Point $P$ lies inside triangle $ABC$ such that the equality $\angle PAC = \angle PBC$ is satisfied (fig. 16). Points $K$ and $L$ are the midpoints of the line segments $CP$ and $AB$, respectively. Let $M$ be the foot of the perpendicular from $P$ to the angle bisector of the angle $ACB$. Prove that the points $K$, $L$ and $M$ are collinear.

17. A convex quadrilateral $ABCD$ with $AB = BC$ is inscribed in a circle (fig. 17). Point $M$ is the midpoint of $AC$ and $K$ is the foot of the perpendicular from $B$ to $CD$. Prove that the lines $BD$ and $KM$ are perpendicular.

18. From a point $P$ lying outside of a circle with center $O$ two tangents $PA$ and $PB$ are drawn (fig. 18). Point $M$ lies on the line segment $AB$. The line passing through $M$ and perpendicular to $OM$ intersects the lines $AP$ and $BP$ at $K$ and $L$, respectively. Prove that $KM = LM$. 
19. An acute-angled triangle $ABC$ is inscribed in a circle $\omega$ (fig. 19). The tangents to the circle $\omega$ at $A$ and $C$ intersect at $F$. The perpendicular bisector of the line segment $AB$ intersects the line $BC$ at $E$. Prove that the lines $FE$ and $AB$ are parallel.

20. Given is a convex quadrilateral $ABCD$ with (fig. 20) $\angle BAC = 44^\circ$, $\angle BCA = 17^\circ$, $\angle CAD = \angle ACD = 29^\circ$. Determine the measure of the angle $ABD$.

21. The incircle of triangle $ABC$ is tangent to the sides $BC$ and $CA$ at points $K$ and $L$, respectively (fig. 21). Let $I$ denote the incenter of triangle $ABC$. The lines $AI$ and $KL$ meet at $P$. Prove that the lines $AP$ and $BP$ are perpendicular.

22. The excircle $\omega$ of triangle $ABC$ is tangent to the side $BC$ at point $D$ and to the line $AC$ at point $E$ (fig. 22). Let $J$ be the center of $\omega$. Denote by $Q$ the foot of the perpendicular from $B$ to $AJ$. Prove that the points $D, E$ and $Q$ are collinear.

23. Let $ABC$ be an acute-angled triangle and let $D$ be the foot of the perpendicular from the point $C$ to the line $AB$ (fig. 23). Points $K$ and $L$ are the feet of the perpendiculars from the points $A$ and $B$, respectively, to the angle-bisector of the angle $ACB$. Point $M$ is the midpoint of the line segment $AB$. Prove that the points $D, K, L, M$ are concyclic.

24. In an acute-angled triangle $ABC$ point $D$ is the foot of the perpendicular from the point $C$ to the line $AB$. (fig. 24). Points $K$ and $L$ are the feet of the perpendiculars from the points $A$ and $B$, respectively, to the angle-bisector of the angle $ACB$. Point $M$ is the midpoint of the line segment $AB$. Prove that the points $D, K, L, M$ are concyclic.
25. Let $ABCD$ be a convex quadrilateral (fig. 25). Prove that there exists an excircle of the quadrilateral $ABCD$ if and only if $AB + BC = AD + DC$.

26. Let $ABCD$ be a concave quadrilateral (fig. 26). Prove that there exists an excircle of the quadrilateral $ABCD$ if and only if $AB + BC = AD + DC$.

27. Given two circles $\omega_1$ and $\omega_2$ (fig. 27). A transversal common tangent $l$ touches the circles $\omega_1$ and $\omega_2$ at points $B$ and $C$, respectively. The line $l$ intersects the external common tangents to the circles $\omega_1$ and $\omega_2$ at points $A$ and $D$. Prove that $AB = CD$.

28. An external common tangent to two circles $\omega_1$ and $\omega_2$ touches the circles $\omega_1$ and $\omega_2$ at $A$ and $B$, respectively (fig. 28). A transversal common tangent to the circles $\omega_1$ and $\omega_2$ intersects their external common tangents at points $A$ and $D$. Prove that $AB = CD$.

29. Two excircles of triangle $ABC$ are tangent to the sides $BC$ and $AC$ of triangle $ABC$ at points $D$ and $E$, respectively (fig. 29). Prove that $AE = BD$.

30. Prove that there exists an incircle of quadrilateral $ABCD$ if and only if the incircles of triangles $ABD$ and $BCD$ are tangent (fig. 30).

31. Let $ABCD$ be a circumscribed quadrilateral (fig. 31). Point $P$ lies on the side $CD$. Prove that there exists a common tangent to the incircles of triangles $ABP$, $BCP$ and $ADP$. 


32. Point \( D \) lies on side \( AB \) of triangle \( ABC \) (fig. 32). The incircles of triangles \( ABC \), \( ADC \) and \( BDC \) are tangent to the line \( AB \) at points \( I \), \( J \) and \( K \), respectively. Prove that \( IJ = DK \).

33. A variable point \( D \) lies on side \( AB \) of a fixed triangle \( ABC \) (fig. 33). The external common tangent of the incircles of triangles \( ADC \) and \( BDC \) (different from the line \( AB \)) intersects the line \( CD \) at point \( E \). Find the locus of the points \( E \), as \( D \) varies on the line segment \( AB \).

34. Points \( P \) and \( Q \) lie on sides \( AB \) and \( AD \) of a convex quadrilateral \( ABCD \) (fig. 34). The lines \( DP \) and \( BQ \) meet at \( S \). Prove that if there exist incircles of quadrilaterals \( APSQ \) and \( BCDS \), then there exists an incircle of quadrilateral \( ABCD \).

35. Let \( D \) be a point lying on side \( AB \) of triangle \( ABC \) such that \( CD = AC \) (fig. 35). The incircle of triangle \( ABC \) is tangent to the sides \( AC \) and \( AB \) at points \( E \) and \( F \), respectively. Let \( I \) denote the incircle of triangle \( BCD \) and let the lines \( AI \) and \( EF \) meet at \( P \). Prove that \( AP = PI \).

36. A convex quadrilateral \( ABCD \) is cut into 9 convex quadrilaterals, as shown on figure 36. Prove that if there exist incircles of the shaded quadrilaterals, then there exists an incircle of quadrilateral \( ABCD \).
Hints and suggestions

Remark: The suggested attempts to problems are not the only ones. Most of the problems can be solved in several ways. Readers are encouraged to find their own ways to solutions before they read the proposed hints.

1. Note that the points $A$, $B$, $D$, $E$ lie on a common circle, whose center is $M$. The equality $\angle DME = 60^\circ$ follows from $\angle EMD = 2\angle EBD$.

2. Let $H$ be the intersection of the altitudes of triangle $ABC$. Observe that the quadrilaterals $AFHE$ and $BFHD$ are cyclic. You will need to note that $\angle CAD = \angle CBE$.

3. Observe that points $C$, $D$, $F$, $P$ are concyclic and that this implies $\angle CPF = 90^\circ$. Next use this to show that $A$, $E$, $P$, $F$ are concyclic. Analogously, show that $A$, $F$, $Q$, $E$ are concyclic and conclude that all the five points lie on a common circle.

4. Let $P$ be the intersection of the line segments $AF$ and $BG$. Observe that the triangles $CGB$ and $CAF$ are congruent and use it to show that the points $A$, $P$, $C$, $G$ are concyclic, and also $B$, $P$, $C$, $F$ are concyclic. Then determine each of the angles $HPA$, $APB$, $BPE$ and conclude that $P$ lies on the line segment $EH$.

5. Let $P$ be the intersection of the line segments $AD$ and $BE$. Observe that the triangles $ACD$ and $ECB$ are congruent and use it to show that the quadrilaterals $AECP$ and $BDCP$ are cyclic. Deduce now that the points $A$, $F$, $B$, $P$ are concyclic. Finally, determine the angles $\angle CPD$, $\angle DPB$ and $\angle BPF$ and conclude that $P$ lies on the line segment $CF$.

6. Let $P$ be the intersection of the line segments $AD$ and $BE$. Observe that the triangles $ACD$ and $ECB$ are congruent and use it to show that the quadrilaterals $ACPE$ and $BDPC$ are cyclic. Deduce now that the points $A$, $F$, $B$, $P$ are concyclic. Determine the angles $\angle APC$ and $\angle APF$ and conclude that $C$ lies on the line $PF$.

7. Observe that the quadrilaterals $AMLC$ and $DLKC$ are cyclic and use it to show that $\angle KLC + \angle ALM = 90^\circ$.

8. Draw a line $l$ parallel to $AB$ through $C$ and let $AD$ and $l$ intersect at $E$. Show that $MBEC$ is a rectangle and that the points $S$, $M$, $B$, $E$ are concyclic. Conclude that the points $B$, $M$, $S$, $C$ are concyclic.

9. Assume that the circumcircle of triangle $BCP$ intersects the line $CM$ at points $C$ and $Q$. Show that the points $A$, $P$, $Q$, $M$ are concyclic. (You will need to consider several cases depending on the different positions of the point $Q$ on the line $CM$.)

10. Let $E$ be the intersection of the circumcircle of triangle $ABC$ with the line $BD$. Use the given equalities to show that triangles $ADE$ and $CDE$ are isosceles.

   Remark: Observe an extra result: $AD = CD = BD$.

11. Let $Q$ be a point such that $APQD$ is a parallelogram. Observe that $BCQP$ is also a parallelogram. Apply Problem 10 for the quadrilateral $DPCQ$.

12. Assume $BS$ and $AD$ meet at $P$. Show that triangles $ABP$ and $BCE$ are congruent. Use it next to observe that $PFCD$ is a rectangle. Observe also that the points $P$, $S$, $C$, $D$ are concyclic.
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13. Let the lines $EP$ and $AB$ meet at $S$. Observe that the quadrilaterals $BQEP$, $CEPD$ and $ADPS$ are cyclic.

14. Let $Q$ be a point such that $BCPQ$ is a parallelogram. Observe that the quadrilateral $ADPQ$ is also a parallelogram and that the points $A$, $P$, $B$, $Q$ are concyclic.

15. Let $Q$ be a point such that $ABPQ$ is a parallelogram. Observe that the quadrilateral $PCDQ$ is also a parallelogram and that the points $A$, $Q$, $P$, $D$ are concyclic.

16. Let $N$ be the reflection of the point $P$ with respect to the line $CM$ and denote by $Q$ the point such that $APBQ$ is a parallelogram. Use Problem 15 to show that $N$ lies on the line $CQ$ and then conclude that $K$, $L$, $M$ are collinear.

17. Observe that the points $B$, $C$, $K$, $M$ are concyclic. Use it show that $\angle BDC + \angle MKD = 90^\circ$.

18. Observe that the quadrilaterals $AKOM$ and $BOML$ are cyclic and use it to show that the triangle $OKL$ is isosceles.

19. Denote by $O$ the circumcenter of triangle $ABC$ (observe that $O$ lies on the perpendicular bisector of $AB$). Show that the points $A$, $O$, $E$, $C$ are concyclic by proving that $\angle AOC = \angle AEC$. Observe also that the points $A$, $O$, $C$, $F$ are concyclic. You should consider moreover the case $AC > BC$.

20. Let the line passing through $D$ and perpendicular to the line $AC$ meet the line $BC$ at $E$. Show that the points $A$, $B$, $E$, $D$ are concyclic.

21. Denote $\alpha = \angle BAI$ and $\beta = \angle ABI$. Observe that the points $I$, $B$, $P$, $K$ are concyclic by computing (in terms of $\alpha$ and $\beta$) the angles $\angle BIP$ and $\angle BKP$. You should also consider the case, where $P$ lies between $K$ and $L$.

22. Denote $\alpha = \angle CBJ$ and $\beta = \angle BCJ$. Using that the quadrilaterals $BJDQ$ and $JDCE$ are cyclic determine the angles $\angle QDJ$ and $\angle JDE$ and conclude that the points $Q$, $D$, $E$ are collinear. Consider also the case, where $Q$ lies between $D$ and $E$.

23. Let $AI$ and $KL$ meet at $P$. Use Problem 21 to observe that the points $A$, $B$, $P$, $D$ are concyclic and that the points $I$, $B$, $P$, $K$ are concyclic. Next show that $PK$ is the angle bisector of angle $APD$ and conclude that $E$ lies on the line $AI$.

24. Let the lines $BL$ and $AC$ meet $P$. Show that the lines $AC$ and $LM$ are parallel. Next use that the points $A$, $D$, $K$, $C$ are concyclic to show that $\angle MLK = \angle MDK$.

25 and 26. Use the same method as in proof of Theorem 5.

27. Set $x = AB$, $y = BC$, $z = CD$. Let an external common tangent to $\omega_1$ and $\omega_2$ touch $\omega_1$ and $\omega_2$ at $P$ and $Q$, respectively. Let the second external common tangent to $\omega_1$ and $\omega_2$ touch $\omega_1$ and $\omega_2$ at $R$ and $S$, respectively. Express $PQ$ and $RS$ in terms of $x$, $y$, $z$ and observe that $PQ = RS$.

28. Use Problem 27.

29. Inscribe a circle in triangle $ABC$ and use Problem 27.

30. Let the incircles of triangles $ABD$ and $BCD$ touch the line segment $BD$ at points $K$ and $L$, respectively. Set $x = BK$, $y = KL$, $z = LD$ and express the sides of quadrilateral $ABCD$ in terms of $x$, $y$ and $z$. 

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31. Let the external common tangent to the incircles of triangles \( ADP \) and \( BCP \) (different from the line \( CD \)) meet \( AP \) and \( BP \) at \( R \) and \( S \), respectively. Use Theorem 3 to show that \( AB + RS = AR + BS \) is equivalent to \( AB + CD = AD + BC \). In a similar way Problem 36 can be solved.

32. Set \( a = BC \), \( b = AC \), \( c = CD \), \( x = AD \), \( y = BD \). Use Theorem 4 to find \( IJ \) and \( DK \).

33. Set \( a = BC \), \( b = AC \), \( c = CD \), \( x = AD \), \( y = BD \). Use Theorem 4 and Problem 27 to determine the length of \( CE \).

34. Observe that the given circles are inscribed in quadrilaterals \( ABSD \) and \( BSDC \). Use Theorem 5 to show that \( AB + CD = AD + BC \).

35. Let the line passing through \( I \) and parallel \( EF \) meet \( AB \) at \( Q \). Show that triangle \( DIQ \) is isosceles observing that \( \angle AFE = \angle IDB \). Next use this to show that \( AF = FQ \) (set \( a = BC \), \( b = AC \), \( c = CD \), \( x = AD \), \( y = BD \) and compute the lengths of \( AF \) and \( FQ \)).

36. Denote the points of tangency as shown on figure 37. Observe that there exists an incircle of the central quadrilateral if and only if \( MN + RQ = ST + PO \). Use this equality to show that \( AB + CD = BC + DA \).